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Solutions for a relativistic string in a uniform static external field

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Abstract. Some exact solutions are obtained for the non-linear coupled partial differential equations obtained by Lund and Regge for a relativistic string in a uniform external field.

1. Introduction

If a moving string is considered as a two-dimensional space–time surface then many physical characteristics of the string are represented by the metric of the surface. Let the metric be given by

$$ds^2 = \cos^2\theta\, d\sigma^2 + \sin^2\theta\, d\tau^2,$$

where σ is a space-like coordinate and τ is a time-like one.

Lund and Regge (1976) have shown that, for a string moving in a uniform static external field, θ satisfies the following equations:

$$\theta_{\tau\tau} - \theta_{\sigma\sigma} + c\sin\theta\cos\theta + (\cos\theta/\sin^3\theta)(\lambda_\tau^2 - \lambda_\sigma^2) = 0 \qquad (1.1a)$$

$$(\lambda_\tau \cot^2\theta)_\tau = (\lambda_\sigma \cot^2\theta)_\sigma \qquad (1.1b)$$

where

$$\theta_\tau \equiv \partial\theta/\partial\tau, \qquad \lambda_\tau \equiv \partial\lambda/\partial\tau, \qquad \theta_{\sigma\sigma} \equiv \partial^2\theta/\partial\sigma^2 \quad \text{and so on.}$$

c is the constant representing the uniform static external field and λ is a function introduced for convenience.

Solutions for (1.1) for the case when θ and λ are both functions of σ have been obtained by Lund and Regge (1976). In this paper we seek solutions of the same equations under the restrictive condition that at least one of λ and θ is a function of a linear combination of τ and σ.

2. Solutions for $\lambda = \lambda(a\tau + b\sigma)$

Because of the Lorentz-invariant character of equations (1.1), it is obvious that if λ is a function of a linear combination of τ and σ then one can, without loss of generality, set $\lambda = \lambda(\tau)$ or $\lambda = \lambda(\sigma)$ or $\lambda = \lambda(\tau - \sigma)$.

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If $\lambda = \lambda(\tau)$, then using equation (1.1*b*) one can set

$$\tan\theta = T\Sigma$$

where (2.1)

$$T = T(\tau), \qquad \Sigma = \Sigma(\sigma)$$

and

$$\lambda_\tau = \pm T^2. \tag{2.2}$$

Using equations (2.1) and (2.2) one can reduce equation (1.1*a*) to

$$\left[\left(\frac{T_{\tau\tau}}{T} - \frac{\Sigma_{\sigma\sigma}}{\Sigma}\right) - \frac{2T^2\Sigma^2}{1+T^2\Sigma^2}\left(\frac{T_\tau^2}{T^2} - \frac{\Sigma_\sigma^2}{\Sigma^2}\right) + \frac{(1+T^2\Sigma^2)^2 + c\Sigma^4}{\Sigma^4}\right] = 0. \tag{2.3}$$

Differentiating equation (2.3) with respect to T and σ, we obtain

$$T_\tau\Sigma_\sigma\left[\left(\frac{T_{\tau\tau}}{T} - \frac{\Sigma_{\sigma\sigma}}{\Sigma}\right) - \frac{T^2\Sigma^2}{1+T^2\Sigma^2}\left(\frac{T_\tau^2}{T^2} - \frac{\Sigma_\sigma^2}{\Sigma^2}\right) + \frac{(1+T^2\Sigma^2)^2}{\Sigma^4}\right] = 0. \tag{2.4}$$

If neither T nor Σ is constant, then from equations (2.3) and (2.4) we obtain

$$T_\tau^2/T^2 - \Sigma_\sigma^2/\Sigma^2 = c(1 + 1/T^2\Sigma^2). \tag{2.5}$$

Differentiating equation (2.5) with respect to τ and σ, we obtain

$$T_\tau\Sigma_\sigma/T^3\Sigma^3 = 0. \tag{2.6}$$

This is impossible unless at least one of T and Σ is constant. If at least one of T and Σ is constant, then we see from equation (2.1) that θ is a function of τ or a function of σ, or a constant.

In a similar way one can show that, if $\lambda = \lambda(\sigma)$, then also θ must be a function of τ or a function of σ, or a constant.

For $\lambda = \lambda(\tau - \sigma)$, it can be checked by direct substitution that no solution exists for equation (1.1).

Therefore, all the solutions of equation (1.1) for which λ is a function of a linear combination of τ and σ are included in the solutions of equation (1.1) for which θ is a function of a linear combination of τ and σ. However we note that for solutions of equation (1.1) λ and θ may be functions of different linear combinations of τ and σ.

3. Solutions for $\theta = \theta(a + b\sigma)$

As before, owing to the Lorentz invariance of equation (1.1), one can set $\theta = \theta(\sigma)$ or $\theta = \theta(\tau)$ or $\theta = \theta(\tau - \sigma)$. Therefore we consider the following cases.

3.1.

$$\theta = \theta(\sigma) \text{ with } \theta_\sigma \neq 0. \tag{3.1}$$

We shall show that for this case

$$\lambda = p\tau + n\int \tan^2\theta \, \mathrm{d}\sigma, \tag{3.2}$$

where p and n are constants. The proof is as follows: for $\lambda_\sigma = 0$ we obtain, from equations (1.1b) and (3.1),

$$\lambda_{\tau\tau} = 0,$$

which is equivalent to equation (3.2) with $n = 0$. For $\lambda_\sigma \neq 0$ we proceed as follows: differentiating equation (1.1a) with respect to τ and using equation (3.1),

$$\lambda_\tau \lambda_{\tau\tau} - \lambda_\sigma \lambda_{\sigma\tau} = 0. \tag{3.3}$$

Also from equation (3.1), one can simplify equation (1.1b) to

$$\lambda_{\tau\tau} - \lambda_{\sigma\sigma} + 2\lambda_\sigma \theta_\sigma / \sin\theta \cos\theta = 0. \tag{3.4}$$

From equations (3.4) and (3.3),

$$\lambda_\sigma \lambda_{\sigma\tau} - \lambda_{\sigma\sigma}\lambda_\tau + 2\lambda_\sigma \lambda_\tau \theta_\sigma / \sin\theta \cos\theta = 0,$$

which can be rewritten as

$$(\lambda_\tau/\lambda_\sigma)_\sigma + 2(\lambda_\tau/\lambda_\sigma)\theta_\sigma / \sin\theta \cos\theta = 0. \tag{3.5}$$

Equation (3.5) can be integrated to give

$$\lambda_\tau = \beta \lambda_\sigma \cot^2\theta, \qquad \text{where } \beta = \beta(\tau). \tag{3.6}$$

Equation (3.6) can be solved for λ to give

$$\lambda = \lambda(y), \qquad \text{where } y = u + v, \qquad u = \int \beta \, \mathrm{d}\tau, \qquad v = \int \tan^2\theta \, \mathrm{d}\sigma. \tag{3.7}$$

Also, from equations (1.1a) and (3.1),

$$\lambda_\tau^2 - \lambda_\sigma^2 = \alpha(\sigma), \tag{3.8}$$

where

$$\alpha = \alpha(\sigma) = (\sin^3\theta / \cos\theta)\theta_{\sigma\sigma} - c \sin^4\theta. \tag{3.9}$$

From equations (3.7) and (3.8),

$$\lambda_y^2(\beta^2 - \tan^4\theta) = \alpha, \tag{3.10}$$

$$\frac{[(\beta^2 - \tan^4\theta)/\alpha]_\sigma}{[(\beta^2 - \tan^4\theta)/\alpha]_\tau} = \frac{(1/\lambda_y^2)_\sigma}{(1/\lambda_y^2)_\tau} = \frac{y_\sigma}{y_\tau} = \frac{\tan^2\theta}{\beta},$$

which on simplication give

$$\frac{-\beta^2 \alpha_\sigma}{\alpha \tan^2\theta} - \frac{\alpha}{\tan^2\theta}\left(\frac{\tan^4\theta}{\alpha}\right)_\sigma = 2\beta_\tau. \tag{3.11}$$

Differentiating equation (3.11) with respect to τ and σ,

$$-2\beta\beta_\tau(\alpha_\sigma/\alpha \tan^2\theta)_\sigma = 0.$$

Therefore either

$$\beta_\tau = 0, \tag{3.12a}$$

or

$$(\alpha_\sigma/\alpha \tan^2\theta)_\sigma = 0. \tag{3.12b}$$

If equation (3.12b) is true, then

$$\alpha_\alpha/\alpha \tan^2\theta = \text{constant}. \qquad (3.13)$$

From equations (3.11) and (3.13)

$$-(\beta^2 M + 2\beta_\tau) = \frac{\alpha}{\tan^2\theta}\left(\frac{\tan^4\theta}{\alpha}\right)_\sigma. \qquad (3.14)$$

Since the left-hand side of equation (3.14) is a function of τ only and the right-hand side is a function of σ only, equation (3.14) can hold only if both are constants.

$$\therefore \quad \frac{\alpha}{\tan^2\theta}\left(\frac{\tan^4\theta}{\alpha}\right)_\sigma = \text{constant}. \qquad (3.15)$$

Then from equations (2.10), (3.13) and (3.15) we obtain three relations giving α and θ as functions of σ, and it can be checked that the three relations cannot be satisfied simultaneously except by making θ a constant, which is impossible. Therefore (3.12a) must hold, i.e.

$$\beta = \text{constant} = m\,(\text{say}). \qquad (3.16)$$

From equations (3.10), (3.11) and (3.16)

$$\lambda_y^2 = \frac{\alpha}{m^2 - \tan^4\theta} = \text{constant} = n^2\ (\text{say}) \qquad (3.17)$$

or $\lambda = mn\tau + n\int \tan^2\theta\, d\sigma$, which reduces to equation (3.2) with $p = mn$. Therefore equation (3.2) holds for both $\lambda_\sigma = 0$ and $\lambda_\sigma \neq 0$. Using equations (3.1) and (3.2) one can solve equation (1.1) for θ to obtain

$$\int \frac{d\theta}{(c\sin^2\theta - p^2/\sin^2\theta + d - n^2/\cos^2\theta)^{1/2}} = \pm\sigma, \qquad (3.18)$$

where p, d and n are constants of integration. It can be checked by direct substitution that equations (3.2) and (3.18) together satisfy equation (1.1).

3.1.1. $\theta = \theta(\sigma)$, $\theta_\sigma \neq 0$, $\lambda_\sigma \neq 0$, $\lambda_\tau \neq 0$. Here solutions are given by equations (3.2) and (3.18) with $p \neq 0$, $n \neq 0$. We note that for given values of p, d and n, if θ is sufficiently close to $k\pi/2$ where k is an integer, then in this case the integral in equation (3.18) becomes imaginary. Therefore there exists k such that

$$k\pi/2 < \theta < (k+1)\pi/2; \qquad (3.19)$$

or, without loss of generality, we can set $k = 0$, i.e. $0 < \theta < \pi/2$.

Setting $u = \sin^2\theta$, equation (3.18) can be rewritten as

$$\int \frac{du}{[-cu^3 + u^2(c-d) + u(p^2 + d - n^2) - p^2]^{1/2}} = \pm\sigma. \qquad (3.20)$$

The condition that u_σ is real further requires that

$$f(u) = -cu^3 + u^2(c-d) + u(p^2 + d - n^2) - p^2 \geqslant 0. \qquad (3.21)$$

From equation (3.19), $0 \leqslant u \leqslant 1$. However, from equation (3.21),

$$f(0) = -p^2 < 0 \qquad \text{and} \qquad f(1) = -n^2 < 0.$$

Therefore there must exist ξ and η such that

$$f(\xi)=0=f(\eta),$$

and $f(u)\geqslant 0$ for $0<\eta<u<\xi<1$. Also from equation (3.21) we note that

$$f(-\infty)>0 \qquad \text{and} \qquad f(0)<0;$$

therefore there exists $\zeta<0$ such that $f(\zeta)=0$. Therefore ξ, η, ζ are the three roots of the equation $f(u)=0$, or equation (3.20) can be rewritten as

$$\int \frac{\mathrm{d}u}{[(\xi-u)(u-\eta)(u-\zeta)]^{1/2}}=\pm\sqrt{c}\sigma, \tag{3.22}$$

where

$$\zeta<0<\eta<u<\xi<1,$$
$$\xi+\eta+\zeta=1-d/c,$$
$$(\xi-1)(1-\eta)(1-\zeta)=-n^2/c,$$
$$\xi\eta\zeta=-p^2/c.$$

From equation (3.22) we see that for u to be real and bounded between 0 and 1, it can only increase from η to ξ and then decrease from ξ to η, and so on as σ increases. Therefore the solutions can be written as follows:

$$u=\sin^2\theta,$$

$$2rI_1+\int_\eta^u \frac{\mathrm{d}u}{[(\xi-u)(u-\eta)(u-\zeta)]^{1/2}}=\sigma \qquad \text{if } 2rI_1\leqslant\sigma\leqslant(2r+1)I_1,$$

$$(2r+1)I_1+\int_u^\xi \frac{\mathrm{d}u}{[(\xi-u)(u-\eta)(u-\zeta)]^{1/2}}=\sigma \qquad \text{if } (2r+1)I_1\leqslant\sigma\leqslant(2r+2)I_2, \tag{3.23}$$

where $I_2=\int_\eta^\xi \frac{\mathrm{d}u}{[(\xi-u)(u-\eta)(u-\zeta)]^{1/2}}$,

$$\lambda=(-c\xi\eta\zeta)^{1/2}+[c(1-\xi)(1-\eta)(1-\zeta)]^{1/2}\int\tan^2\theta\,\mathrm{d}\theta,$$

r is any integer and ξ, η and ζ are constants of integration such that

$$\zeta<0<\eta<\xi<1.$$

3.1.2. $\theta=\theta(\sigma)$, $\theta_\sigma\neq 0$, $\lambda_\tau=0$, $\lambda_\sigma\neq 0$. Here, solutions are given by equations (3.2) and (3.18) with $p=0$, $n\neq 0$. We note that if, for given values of p, d and n, θ is sufficiently close to $(2k+1)\pi/2$ where k is an integer, then in this case the integral in equation (3.18) becomes imaginary. Therefore there exists k such that

$$\tfrac{1}{2}(2k-1)\pi<\theta<\tfrac{1}{2}(2k+1)\pi.$$

Without loss of generality we can set $k=0$, i.e.

$$-\pi/2<\theta<\pi/2.$$

Therefore equation (3.18) reduces to

$$\int \frac{\mathrm{d}v}{(g(u))^{1/2}}=\pm\sigma,$$

where $v = \sin\theta$, and as before $u = \sin^2\theta$,

$$g(u) = -cu^2 + (c-d)u + (d-n^2). \tag{3.24}$$

From equation (3.24), $g(1) = -n^2 < 0$. Since $g(u)$ must be positive for some value of u between 0 and 1, there exists ν such that

$$0 < \nu < 1 \tag{3.25}$$

and $g(\nu) = 0$. However, two roots of a quadratic can be either both real or both complex. Therefore two roots of $g(u) = 0$ must be real, or there exists μ such that

$$g(u) = -c(u-\mu)(u-\nu). \tag{3.26}$$

From equations (3.24) and (3.26)

$$(1-\mu)(1-\nu) = n^2/c^2. \tag{3.27}$$

From equations (3.25) and (3.27)

$$\mu < 1.$$

3.1.2(a) $\mu < 0$. Here the condition that θ and σ are real requires that as σ increases v oscillates between $-\sqrt{\nu}$ and $\sqrt{\nu}$. Therefore the solutions are

$$2rI_2 + \frac{1}{\sqrt{c}}\int_{-\sqrt{\nu}}^{v} \frac{\mathrm{d}v}{[(v^2-\mu)(\nu-v^2)]^{1/2}} = \sigma \qquad \text{if } 2rI_2 < \sigma < (2r+1)I_2,$$

$$(2r+1)I_2 + \frac{1}{\sqrt{c}}\int_{v}^{\sqrt{\nu}} \frac{\mathrm{d}v}{[(v^2-\mu)(\nu-v^2)]^{1/2}} = \sigma \qquad \text{if } (2r+1)I_2 < \sigma < (2r+2)I_2, \tag{3.28}$$

$$v = \sin\theta, \qquad I_2 = \frac{1}{\sqrt{c}}\int_{-\sqrt{\nu}}^{\sqrt{\nu}} \frac{\mathrm{d}v}{[(v^2-\mu)(\nu-v^2)]^{1/2}}, \qquad \lambda = c(1-\mu)(1-\nu)\int \tan^2\theta\,\mathrm{d}\sigma,$$

r is any integer and μ, ν are constants of integration such that

$$\mu < 0 < \nu < 1.$$

3.1.2(b) $\mu = 0$. Here the integration can be explicitly done.

$$\theta = \sin^{-1}(\sqrt{\nu}\,\mathrm{sech}(\sigma\sqrt{\nu})),$$

$$\lambda = (c-\nu)\int \tan^2\theta\,\mathrm{d}\sigma, \tag{3.29}$$

where ν is a constant of integration such that

$$0 < \nu < 1.$$

3.1.2(c) $0 < \mu < 1$. Since μ and ν are both between 0 and 1, without loss of generality we can take $\mu < \nu$. Further, since equation (3.30) becomes imaginary if $\mu = \nu$, we obtain

$\mu < \nu$. v can now oscillate between $\sqrt{\mu}$ and $\sqrt{\nu}$ as σ increases, and the solutions are

$$2rI_3 + \frac{1}{\sqrt{c}}\int_{\sqrt{\mu}}^{v} \frac{dv}{[(v^2-\mu)(\nu-v^2)]^{1/2}} = \sigma \qquad \text{if } 2rI_3 < \sigma < (2r+1)I_3,$$

$$(2r+1)I_3 + \frac{1}{\sqrt{c}}\int_{v}^{\sqrt{\nu}} \frac{dv}{[(v^2-\mu)(\nu-v^2)]^{1/2}} = \sigma \qquad \text{if } (2r+1)I_3 < \sigma < (2r+2)I_3$$

$$I_3 = \frac{1}{\sqrt{c}}\int_{\sqrt{\mu}}^{\sqrt{\nu}} \frac{dv}{[(v^2-\mu)(\nu-v^2)]^{1/2}},$$

$$\lambda = c(1-\mu)(1+\nu)\int \tan^2 d\sigma, \tag{3.30}$$

μ, ν are constants of integration such that

$$0 < \mu < \nu < 1,$$

and r is any integer.

3.1.3. $\theta = \theta(\sigma)$, $\theta_\sigma \neq 0$, $\lambda_\tau \neq 0$, $\lambda_\sigma = 0$. Hence, solutions are given by equations (3.2) and (3.18) with $p \neq 0$, $n = 0$. In this case we note that if, for any given value of p, d and n, θ is sufficiently close to $k\pi$ where k is an integer, then the integral in equation (3.18) becomes imaginary. Therefore there exists k such that

$$k\pi < 0 < (k+1)\pi.$$

Without loss of generality we can set $k = 0$

$$\text{or} \qquad 0 < \theta < \pi$$

or, setting $z = \cos\theta$, equation (3.18) reduces to

$$-\int \frac{dz}{(h(z^2))^{1/2}} = \pm\sigma,$$

where

$$h(z^2) \equiv cz^4 - z^2(2c+d) + (c+d-p^2). \tag{3.31}$$

Also

$$h(1) = -p^2 < 0. \tag{3.32}$$

Using arguments similar to case 3.1.2,

$$h(z^2) = c(z^2-\gamma)(z^2-\delta) \tag{3.33}$$

where

$$0 < \gamma < 1, \tag{3.34}$$

or, from equations (3.32) and (3.33),

$$(1-\gamma)(1-\delta) < 0. \tag{3.35}$$

From equations (3.34) and (3.35), $\delta > 1$. Now the solutions can be written down as follows:

$$2rI_4 + \frac{1}{\sqrt{c}} \int_{\sqrt{\gamma}}^{z} \frac{dz}{[(z^2-\gamma)(z^2-\delta)]^{1/2}} = \sigma \qquad \text{if } 2rI_4 < \sigma < (2r+1)I_4,$$

$$(2r+1)I_4 + \frac{1}{\sqrt{c}} \int_{z}^{1} \frac{dz}{[(z^2-\gamma)(z^2-\delta)]^{1/2}} = \sigma \qquad \text{if } (2r+1)I_4 < \sigma < (2r+2)I_4, \tag{3.36}$$

$z = \cos\theta$,

$$I_4 = \int_{\sqrt{\gamma}}^{1} \frac{dz}{[(z^2-\gamma)(z^2-\delta)]^{1/2}}.$$

γ, δ are constants of integration such that

$$0 < \gamma < 1 < \delta.$$

3.2.

$$\theta = \mathrm{O}(\tau) \text{ with } \theta_\tau \neq 0. \tag{3.37}$$

Proceeding as in case 3.1, solutions can be obtained for this case too. However, solutions for $\theta = \theta(\tau)$ can also be obtained from solutions for $\theta = \mathrm{O}(\sigma)$ by using a transformation as follows.

From equation (1.1*b*) there exists χ_σ such that

$$\lambda_\tau \cot^2\theta = \chi_\sigma, \qquad \lambda_\sigma \cot^2\theta = \chi_\sigma. \tag{3.38}$$

We note that equations (1.1) remain invariant under the transformation

$$\lambda' = \chi, \qquad \tau' = \sigma, \qquad \sigma' = \tau. \tag{3.39}$$

This reduces solutions for $\theta = \theta(\sigma)$ to solutions for $\theta = \theta(\tau)$.

3.3.

$$\theta = \theta(\tau - \sigma). \tag{3.40}$$

Set

$$u = \tau + \sigma, \qquad v = \tau - \sigma. \tag{3.41}$$

Using equation (3.40), equations (1.1) reduce to

$$c \sin\theta \cos\theta + (4\cos\theta/\sin^3\theta)\lambda_u\lambda_v = 0, \tag{3.42a}$$

$$\lambda_{uv} - \lambda_u\lambda_v/\sin\theta\cos\theta = 0. \tag{3.42b}$$

Dividing by λ_u, equation (3.42*b*) can be rewritten as

$$(1/\lambda_u)\lambda_{uv} = \theta_v/\sin\theta\cos\theta$$

which is equivalent to

$$\lambda_u = \gamma_u \tan\theta, \qquad \text{where } \gamma = \gamma(u) \text{ and } \gamma_u = d\gamma/du. \tag{3.42b)'$$

Equation (3.42*b*)′ is equivalent to

$$\lambda = \gamma \tan\theta + \delta, \qquad \text{where } \gamma = \gamma(u), \delta = \delta(v). \tag{3.42b)''$$

Using equation (3.42*b*)″, we see that equation (3.42*a*) is equivalent to

$$\gamma\gamma_u + \gamma_u\delta_v \cos^2\theta/\theta_v = -c \sin^3\theta \cos^3\theta/4\theta_v.$$

Differentiating equation (3.42*a*)′ with respect to u and v

$$\gamma_{uu}(\delta_v \cos^2\theta/\theta_v)_v = 0,$$

from which we can see that equation (3.42*a*)′ can hold only if

$$\delta_v \cos^2\theta/\theta_v = M, \qquad -c \sin^3\theta \cos^3\theta/4\theta_v = P, \tag{3.43}$$

where M and P are constants. Substituting equation (3.43) into equation (3.42*a*)′ we see that equation (3.42), subject to equation (3.40), is solved as follows.

4. Conclusion

Summarily, for all the solutions of equation (1.1), if λ is a function of a linear combination of τ and σ, then so is θ. For solutions of equation (1.1), if θ is a function of a linear combination of τ and σ, then by a suitable linear transformation we can have $\theta = \theta(\sigma)$ or $\theta = \theta(\tau)$ or $\theta = (\tau - \sigma)$ or $\theta =$ constant. Solutions for $\theta = \theta(\sigma)$ are given by equations (3.23), (3.28), (3.29), (3.30) and (3.36). Solutions for $\theta = \theta(\tau)$ can be obtained from the solutions for $\theta = \theta(\sigma)$ by using the transformation defined through equations (3.38) and (3.39). Solutions for $\theta = \theta(\tau - \sigma)$ including $\theta =$ constant are given by equation (3.43).

Of the various solutions shown here, (3.28), (3.29) and (3.36) pass through singular points where the determinant of the metric tensor vanishes. On the other hand, solutions (3.23), (3.30) and (3.43) do not pass through any singular point. For equation (3.43) the range of θ is the open interval $(0, \pi/2)$. For each of equations (3.23) and (3.30) the range of θ is some closed interval $[\theta_{\min}, \theta_{\max}]$ such that $0 < \theta_{\min} < \theta_{\max} < \pi/2$. In a similar way, singular and non-singular solutions with $\theta = \theta(\tau)$ can be indicated.

It can also be noted that another class of solutions of the same equations has been studied by the author (Ray 1978).

References

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